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Hypothesis Testing & Confidence Intervals

Hypothesis Testing & Confidence Intervals

- **Applied Statistics:**
  - **Simple Random Sample**
  - **Sampling error** is the difference between a sample statistic (the mean, variance, or standard deviation of the sample) and its corresponding population parameter (the true mean, variance, or standard deviation of the population).

    Sampling error of the mean = sample mean – population mean = $\bar{x} - \mu$

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Mean & Variance of the Sample Average:

- The **sampling distribution** of the sample statistic is a probability distribution of all possible sample statistics computed from a set of equal-size samples that were randomly drawn from the same population.

- For n observations:

  Expected value of sample average is: $E(\bar{x}) = \mu_x$

- For n observations:

  Variance of sample average is: $Var(\bar{x}) = \frac{\sigma_x^2}{n}$

  Standards deviation of sample average = $\frac{\sigma_x}{\sqrt{n}}$ also called standard error of sample.

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- **Population & Sample Mean:**
  - Population Mean: $\mu = \frac{\sum_{i=1}^{N} X_i}{N}$
  - Sample Mean: $\bar{X} = \frac{\sum_{i=1}^{n} X_i}{n}$

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- **Population & Sample Variance:**
  - Population Variance: $$\sigma^2 = \frac{\sum_{i=1}^{N}(X_i - \mu)^2}{N}$$
  - Sample Variance: $$s^2 = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}$$
  - Sample Standard Deviation: $$s = \sqrt{\frac{\sum_{i=1}^{n}(X_i - \bar{X})^2}{n-1}}$$

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APPLIED STATISTICS:

Example: Population variance,
Assume the following 5-years annualized total returns represents all of the managers at a small investment from (30%, 12%, 25%, 20%, 23%). What is the population variance & standard deviation of these return?

Example : Sample variance
Assume that the 5-years annualized total returns for the five investment managers used in the preceding examples represent only a sample of the managers at a large investment firm. What is the sample variance & sample standard deviation of these returns?

Example: Standard errors of sample mean (known population variance)
The mean hourly wage for Iowa farm workers is $13.50 with a population standard deviation of $2.90. calculated and interpret the standard error of the sample mean for a sample size of 30.

Example: Standard error of sample mean (unknown population variance)
Suppose a sample contains the past 30 monthly returns for Mc-Creary, Inc. the mean return is 2% and the sample standard deviation is 20%. Calculate and interpret the standard error of the sample mean.

Example: Standard error of sample means (unknown population variance)
Continuing with our example, suppose that instead of a sample size of 30, we take a sample of the past 200 monthly returns for McCreary, Inc. In order to highlight the effect of a sample size on the sample standard error, let's assume that the mean return and standard deviation of this large sample remain at 2% and 20%, respectively. Now, calculate the standard error of the sample mean for the 200 – return sample.

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- **Population & Sample Covariance:**
  - Population Covariance: $$Cov_{X,Y} = \frac{\sum_{i=1}^{N}(X_i - \mu_X)(Y_i - \mu_Y)}{N}$$
  - Sample Covariance: $$Cov_{X,Y} = \frac{\sum_{i=1}^{n}(X_i - \bar{X})(Y_i - \bar{Y})}{n-1}$$

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- **CONFIDENCE INTERVALS:**
  - Confidence intervals :
    Point estimate ± (reliability factor x standard error)

  - If the population has a **normal distribution with a known variance**, a confidence interval for the population mean can be calculated as: $\bar{x} \pm Z_{\alpha/2} \frac{\sigma}{\sqrt{n}}$

Where: $\bar{x}$ = point estimate of the population mean (sample mean)

$Z_{\propto/2}$ =reliability factor

$\frac{\sigma}{\sqrt{n}}$ =the standard error of the sample mean where is the known standard deviation of the population, and n is the sample size.

α is the level of significance & (1- α) is referred to as the degree of confidence.

The most commonly used standard normal distribution reliability factors are:

$Z_{\propto/2}$ =1.65 for 90% confidence interval (the significance level is 10%, 5% in each tail).

$Z_{\propto/2}$ =1.96 for 95% confidence interval (the significance level is 5%, 2.55 in each tail).

$Z_{\propto/2}$ =2.58 for 99% confidence intervals (the significance level is 1%, 0.5% in each tail).

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- **CONFIDENCE INTERVALS:**

Example: Confidence interval

Consider a practice exam that was administered to 36 FRM Part I candidates. The mean scores on this practice exam was 80. Assuming a population standard deviation equal to 15, construct and interpret a 99% confidence interval for the mean score on the practice exam for 36 candidates. Notice that in this exam[le the population standard deviation is known, so we don't have to estimate it:

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CONFIDENCE INTERVALS:

Confidence Intervals for a Population Mean: Normal With Unknown Variance

$$\bar{x} \pm t_{\alpha/2} \frac{\sigma}{\sqrt{n}}$$

Where: $t_{\alpha/2}$ =the t-reliability factor (i.e., t-statistic or critical t-value) corresponding to a t-distributed random variable with n- 1 degrees of freedom.

Example: Confidence intervals

Construct a 95% confidence interval for the mean monthly return, if mean monthly return is 2%, sample standard deviation is 20% & sample size is 30.

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CONFIDENCE INTERVALS:

Confidence Intervals for a Population Mean: Non-normal With Unknown Variance

If the distribution is non-normal, but known population variance with the large sample size → z-statistic can be used

If the distribution is non-normal, but unknown population variance with the large sample size → t-statistic can be used

If the distribution is non-normal & with small sample size → no confidence interval can be created.

Criterion for Selecting the Appropriate test Statistic.

When sampling from a:	*Test Statistic*	
	Small Sample	*Large Sample*
	(n < 30)	*(n> 30)*
Normal distribution with known variance	*z-statistic*	*z-statistic*
Normal distribution with unknown variance	*t-statistic*	*t-statistic**
Normal distribution with known variance	*not available*	*z-statistic*
Normal distribution with unknown variance	*not available*	*t-statistic**

*The z-statistic is theoretically acceptable here, but use of the t-statistic is more conservative.

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➢ HYPOTHESIS TESTING:

➢Hypothesis testing is the statistical assessment of a statement or idea regarding a population.

Hypothesis Testing Procedure

1. State the hypothesis
2. Select the appropriate test statistic
3. Specify the level of significances
4. Select the decision rule regarding the hypothesis
5. Collect the sample and calculate the sample statistics
6. Make a decision regarding the hypothesis
7. Make a decision based on the results of the test

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THE NULL HYPOTHESIS AND ALTERNATIVE HYPOTHESIS:

- The null hypothesis, designated H_0, is the hypothesis the researcher wants to reject.
- The null hypothesis for the population mean can be stated as:

$$H_0: \mu = \mu_0, H_0: \mu \leq \mu_0, \ and \ H_0: \mu \geq \mu_0,$$

$$where \ \mu \ is \ the \ population \ mean \ and \ \mu_0 is \ the \ hypothesised \ value \ of \ population \ mean.$$

- The alternative hypothesis, designed H_A, is what is concluded if there is sufficient evidence to reject the null hypothesis.

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THE NULL HYPOTHESIS AND ALTERNATIVE HYPOTHESIS:

- Hypothesis testing involves two statistics:
  - the test statistic, &
  - the critical value of the test statistic.

- Test statistic = $\dfrac{\textit{sample statistic} - \textit{hypothesized value}}{\textit{standard error of the sample statistic}}$

- Depending on the characteristics of the sample and the population, a test statistic may follow the t-distribution, the z-distribution (standard normal distribution), the chi-squared distribution, and the F-distribution.

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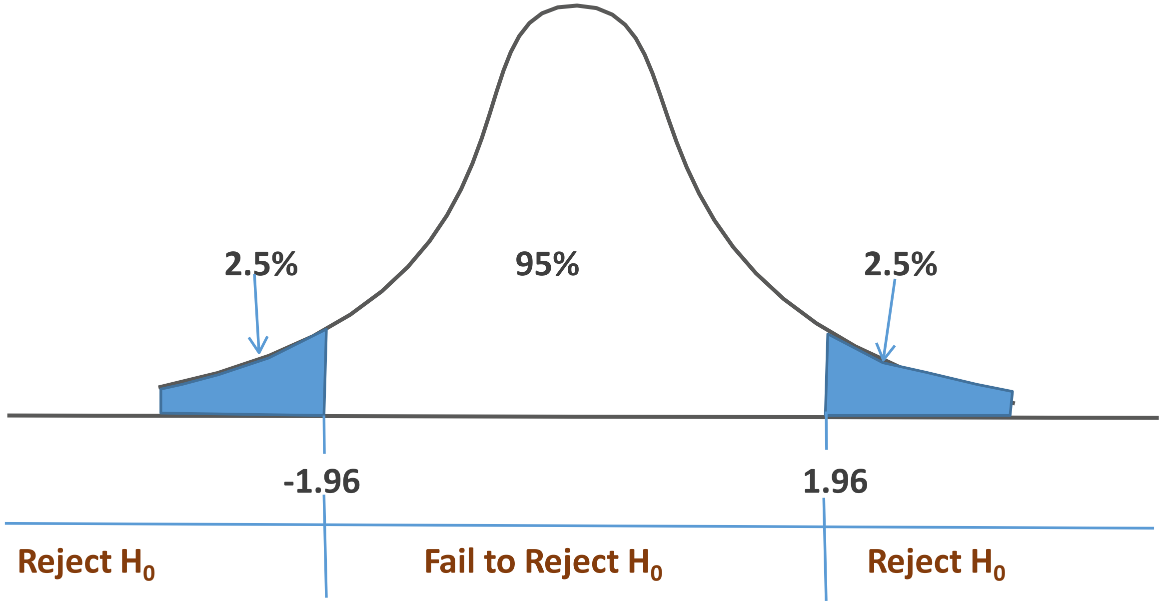
ONE TAILED AND TWO TAILED TEST OF HYPOTHESES:

- A two-tailed test for the population mean may be structured as:

$$H_0: \mu = \mu_0 \ versus \ H_A: \mu \neq \mu_0$$

- The general decision rule for a two tail test is:

  Reject H_0 if: Test statistic > upper critical value or
  Test statistic < lower critical value



Hypothesis Testing & Confidence Intervals

➢ **ONE TAILED AND TWO TAILED TEST OF HYPOTHESES:**

Example: Two-tailed test

A research has gathered data on the daily returns on a portfolio of call options over's a recent 250-day period. The mean daily return has been 0.1%, and the sample standard deviation of daily portfolio returns is 0.25%.the researcher believers the mean daily portfolio return is not equal to zero. Construct a hypothesis test of the researcher's belief.

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ONE TAILED AND TWO TAILED TEST OF HYPOTHESES:

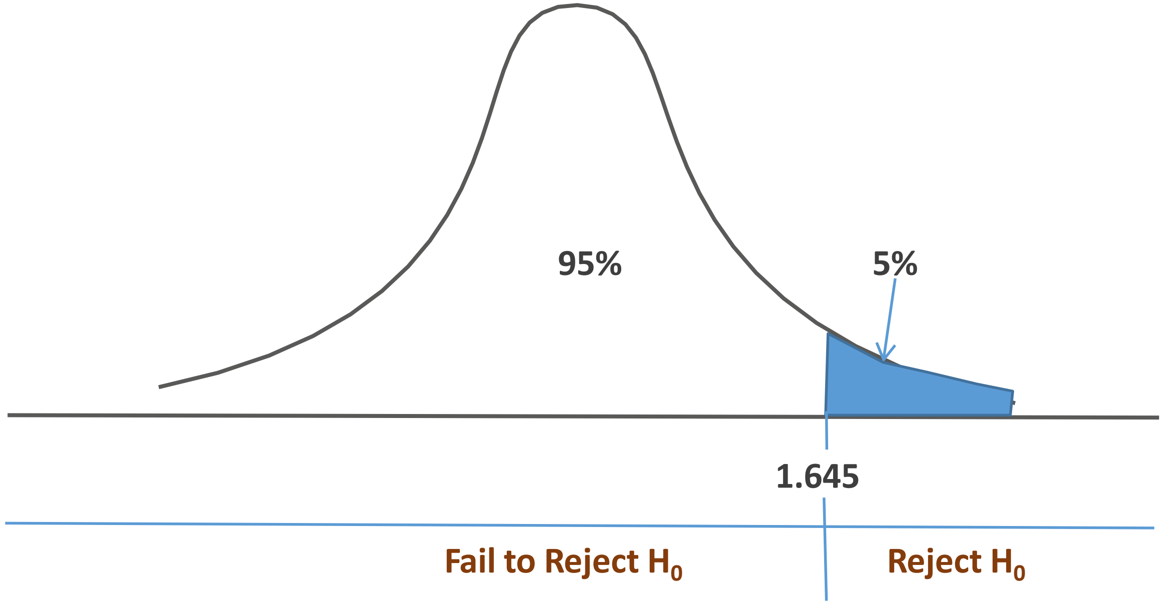
- For a one-tailed hypothesis test of the population mean, the null and alternative hypothesis are either:

$$Upper\ tail: H_0: \mu \leq \mu_0 versus\ H_A: \mu > \mu_{0,}\ or$$

$$Lower\ tail: H_0: \mu \geq \mu_0, varsus H_A: \mu < \mu_{0,}$$

- The general decision rule for a one tail test is:

Reject H_0 if: Test statistic > upper critical value or
Test statistic < lower critical value



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➢ ONE TAILED AND TWO TAILED TEST OF HYPOTHESES:

Example: One-tailed test

Perform a z-test using the option portfolio data from the previous examples to test the belief that option returns are positive.

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- **TYPES I AND TYPE II ERRORS:**
  - When drawing inferences from a hypothesis test, there are two types of error.
    - Type I error: the rejection of the null hypothesis when it is actually true.
    - Type II error: the failure to reject the null hypothesis when it is actually false.
  - The significance level α is the probability of making a Type I error.

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The Power of a Test:

- the power of a test is the probability of correctly rejection the null hypothesis when it is false.
- The power of a test is actually one minus the probability of making a Type II error, or 1 – P(Type II error).

	True Condition	
Decision	*H_0 is true*	*H_0 is false*
Do not reject H_0	*correct decision*	*incorrect decision Type II error*
Reject H_0	*Incorrect decision Type I error Significance level, α , = P (Type I error).*	*Correct decision Power of the test = 1 – P (Type II error)*

Hypothesis Testing & Confidence Intervals

- **THE RELATION BETWEEN CONFIDENCE INTERVALS AND HYPOTHESIS TESTS**:
  - A confidence interval is a range of values within which the research believes the true population parameter may lie.
  - A confidence interval is determined as:

$$\left\{\left[\begin{matrix}\text{sample}\\\text{statistic}\end{matrix} - \begin{pmatrix}\text{critical}\\\text{value}\end{pmatrix}\begin{pmatrix}\text{standard}\\\text{error}\end{pmatrix}\right] \leq \begin{matrix}\text{population}\\\text{parameter}\end{matrix} \leq \left[\begin{matrix}\text{sample}\\\text{statistic}\end{matrix} + \begin{pmatrix}\text{critical}\\\text{value}\end{pmatrix}\begin{pmatrix}\text{standard}\\\text{error}\end{pmatrix}\right]\right\}$$

Example: Confidence interval

Using option portfolio data from the previous example, construct a 95% confidence interval for the population mean daily return over the 250- day sample period. Use a z-distribution. Decide if the hypothesis should be rejected.

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- **Statistical Significance Vs. Economic Significance:**

  - Statistical significance does not necessarily imply economic significance.
  - Factors involved:
    - transaction costs
    - Taxes
    - Additional risk

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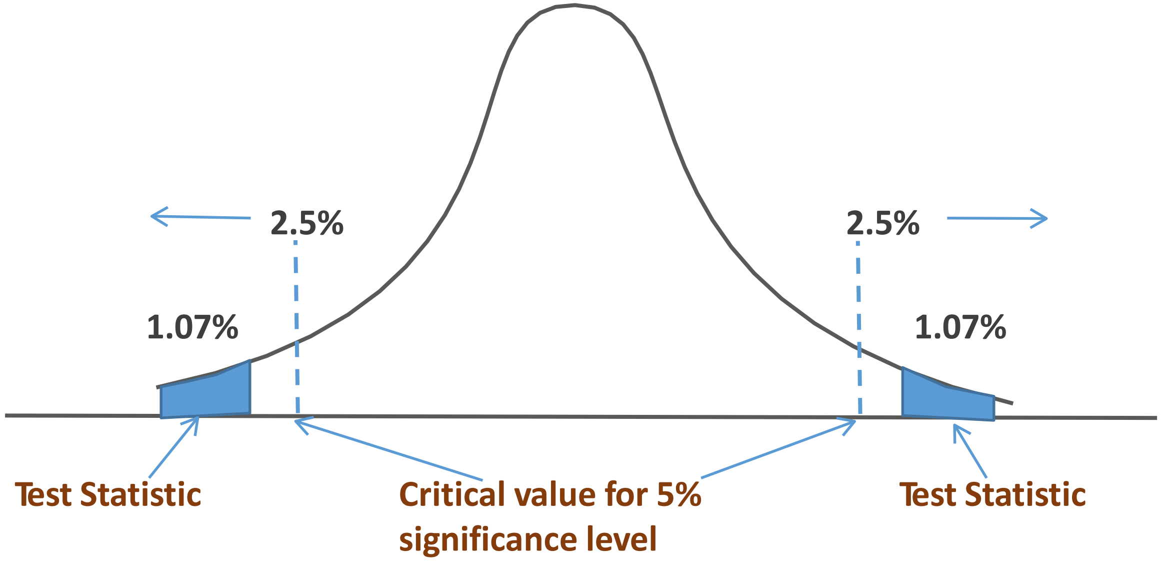
- **THE p-VALUE:**

  - The p-values is the probability of obtaining a test statistic that would lead to a rejection of the null hypothesis, assuming the null hypothesis is true.
  - It is the smallest level of significance for which the null hypothesis can be rejected.
  - For one-tailed test, the p-value is the probability that lies above the computed test statistic for upper tail test or below the computed test statistic for lower tail tests.
  - For two-tailed test, the p-valued is the probability that lies above the positive value of the computed test statistic plus the probability that lies below the negative value of the computed test statistic.

Two-Tailed Hypothesis Test with p-Values= 2 x 1.07=2.14%



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- **THE t-TEST:**
  - A t-statistic for hypothesis testing with n – 1 degree of freedom is computed as:

$$t_{n-1} = \frac{\bar{x} - \mu_0}{s/\sqrt{n}}$$

- **THE Z-TEST:**
  - The z-statistic for a hypothesis test for a population mean is computed as:

$$z - \text{statistic} = \frac{\bar{x} - \mu_0}{\sigma/\sqrt{n}}$$

Hypothesis Testing & Confidence Intervals

Example: z-test or t-test?

Referring to our previous option portfolio mean return problem once more, determine which test statistic (z or t) should be used and the difference in the likelihood of rejecting a true null with each distribution.

Example: The z-test

When your company's gizmo machine is working property, the mean length of gizmo is 2.5 inches. However, from time to time the machine gets out of alignment and produces gizmos that are either too long or too short. When this happens, production is stopped and the machine is adjusted. To check the machine, the quality control department takes a gizmo sample each day. Today, a random sample of 49 gizmos showed a mean length of 2.49 inches. The population standard deviation is known to be 0.021 inches. Using a 5% significance level, determine if the machines should be shut down adjusted.

Hypothesis Testing & Confidence Intervals

THE CHI – SQUARED TEST:

- The chi-squared test is used for hypothesis tests concerning the variance of a normally distributed population.
- The hypotheses for a two-tailed test of a single population variance are structured as:

$$H_0: \sigma^2 = \sigma_0^2 \text{ versus } H_A: \sigma^2 \neq \sigma_0^2$$

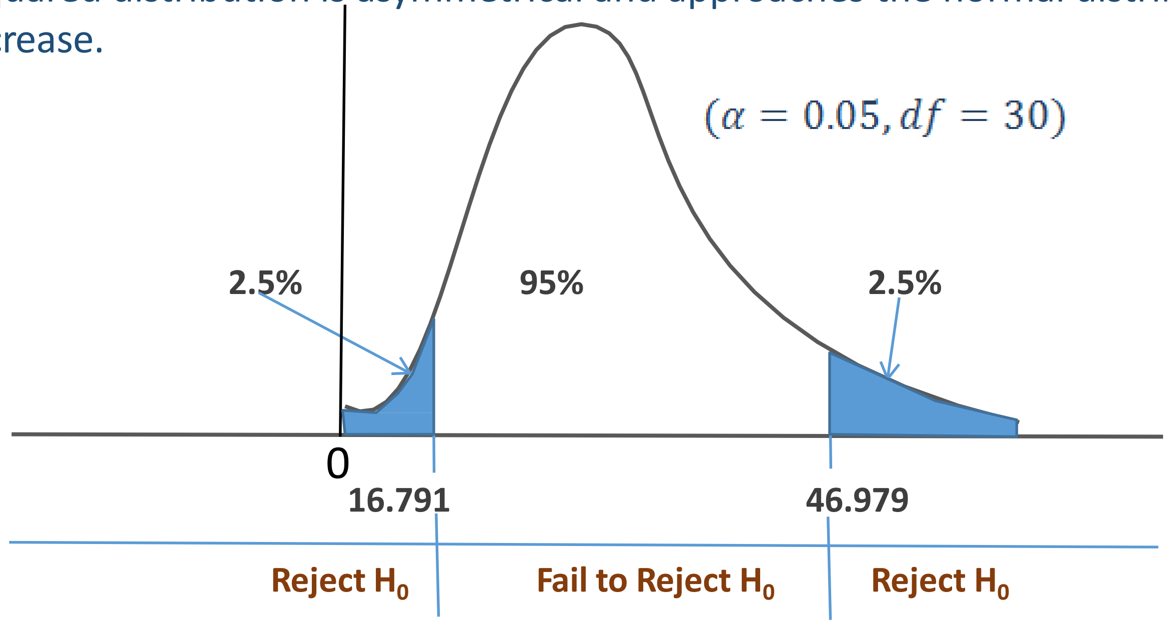
[redacted]The hypotheses for one-tailed tests are structured as:

$$H_0: \sigma^2 \leq \sigma_0^2 \text{ versus } H_A: \sigma^2 > \sigma_0^2$$

$$H_0: \sigma^2 \geq \sigma_0^2 \text{ versus } H_A: \sigma^2 < \sigma_0^2$$

[redacted]The chi-squared distribution test statistics is denoted as χ^2

- The chi-squared distribution is asymmetrical and approaches the normal distribution in shape as the degrees of freedom increase.



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- **THE CHI – SQUARED TEST:**
  - The chi-squared test statistic, χ^2, with n -1 degrees of freedom, is computed as:

$$\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma_0^2}$$

Example: Chi-squared test for a single population variance

Historically, high-Return Equity Fund has advertised that its monthly returns have a standard deviation equal to 4%. This was based on estimates from the 1990- 1998 period High-Return wants to verify whether this claim still adequately describe she standard deviation of the fund’s returns. High-Return collected monthly returns for the 24-month period between 1998 an d2000 and measured a standard deviation of monthly returns of 3.8%. Determine if the more recent standard deviation is different from the advertised standard deviation.

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THE F-TEST:

- The hypotheses concerned with the equality of the variances of *two populations* are tested with an F-distribution test statistic.
- The F-test is used under the assumption that the population from which samples are drawn are normally distributed and that the samples are independent.
- The F-distribution is right-skewed and is truncated at zero on the left-hand side.
- The shape of the F-distribution is determined by two separate degree of freedom, the numerator degrees of freedom, df_1, and the denominator degree of freedom, df_2 .
- The hypotheses for the two-tailed F-test of difference in the variances can be structured as:

$$H_0: \sigma_1^2 = \sigma_2^2 \text{ versus } H_A: \sigma_1^2 \neq \sigma_2^2$$

- And the one-sided test structures can be specified as

$$H_0: \sigma_1^2 \leq \sigma_2^2 \text{ versus } H_A: \sigma_1^2 > \sigma_2^2$$

$$H_0: \sigma_1^2 \geq \sigma_2^2 \text{ versus } H_A: \sigma_1^2 < \sigma_2^2$$

- The test statistic for the F-test is the ratio of the sample variances. The F-statistic is computed as:

$$F = \frac{s_1^2}{s_2^2}$$

- n_1 -1 and n_2 -1 are the degrees of freedom of two samples.

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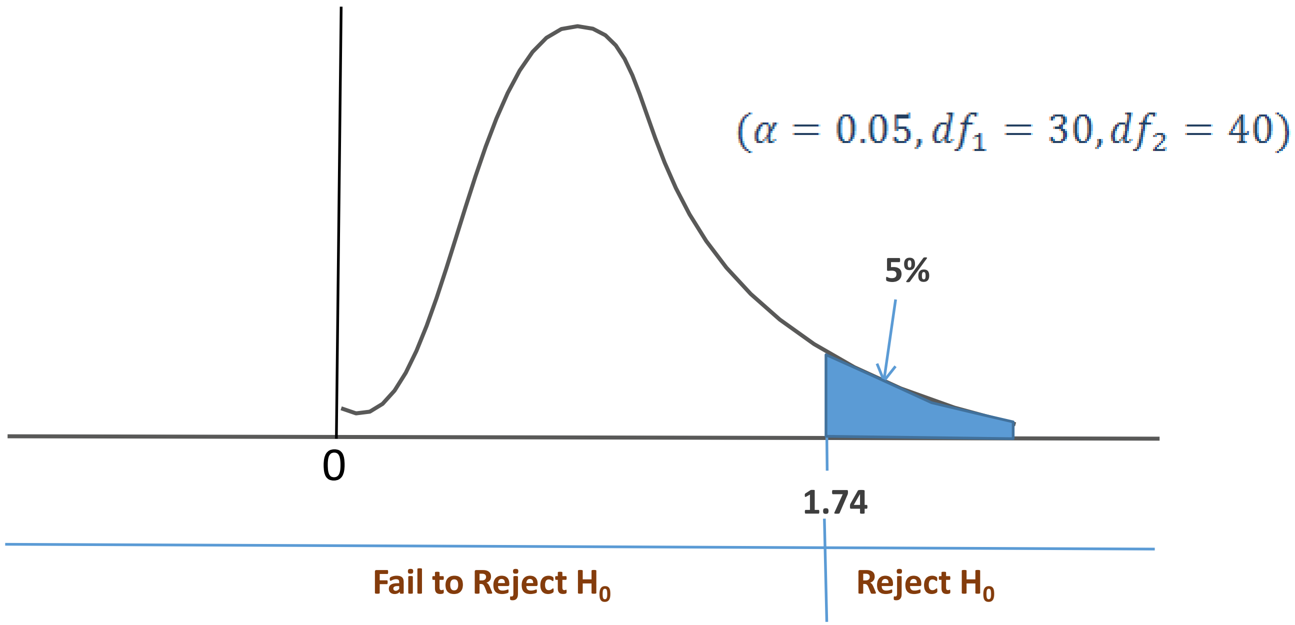
➢ **THE F-TEST:**

Example: F-test for equal variance

Annie Cower is examining the earning for two different industries. Cower suspects that the earnings of the textile industry are more divergent than those of the paper industry. To confirm this suspicion, Cower has looked at a sample of 31 textile manufactures and a sample of 4` paper companies. She measured the sample standard deviation of earning across the textile industry to be \$4.30 and that of the paper industry companies to be \$3.80. Determine if the earnings of the textile industry have greater standard deviation than those of the paper industry.



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- CHEBYSHEV'S INEQUALITY:
  - Chebyshev's inequality states that for any set of observations, whether sample or population data and regardless of the shape of the distribution, the percentage of the observations that lie within k standard deviations of the mean is at least $1 - 1/k^2$ for all $k > 1$.
  - Example: Chebyshev's inequality
  - What is the minimum percentage of any distribution that will lie within +- 2 standard deviation s of the mean?

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- **BACKTESTING:**

  - The process of back testing involves comparing expected outcomes against actual data.

Example: Calculating the number of exceedances

Assume that the value at risk (VaR) of a portfolio, at a 95% confidence interval, is $100 million. Also assume that given a 100-day trading period, the actual number of daily losses exceeding $100 million occurred eight times. Is this VaR model underestimating the actual level of risk?

Thanks
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