/02 Distributions

Objective

- Distinguish the key properties among the following distributions: uniform distribution, Bernoulli distribution, Binomial distribution, Poisson distribution, normal distribution, lognormal distribution, Chi-squared distribution, Student's t, and F-distributions, and identify common occurrences of each distribution.
- Describe the central limit theorem and the implications it has when combining independent and identically distributed (i.i.d.) random variables.
- Describe i.i.d. random variables and the implications of the i.i.d. assumption when combining random variables.
- Describe a mixture distribution and explain the creation and characteristics of mixture distributions.

Distributions

- **PARAMETRIC AND NONPARAMETRIC DISTRIBUTIONS:**
  - Probability distributions are classified into two categories: parametric and nonparametric.
  - Parametric distribution :
    - assume normal distribution of population
    - can be described by using a mathematical function
    - make restrictive assumptions
  - Nonparametric distributions :
    - historical distribution
    - cannot be described by using a mathematical function
    - do not make restrictive assumptions, instead fit the data perfectly

Distributions

➢ THE UNIFORM DISTRIBUTION:

A continuous uniform distribution may be described as follows:

➢For all $a \leq x_1 \leq x_2 \leq b$ (i.e., for all x_1 and x_2 between the boundaries a and b).

➢P(X < a or X > b) = 0 (i.e., the probability of X outside the boundaries is zero).

➢$P(x_1 \leq X x_2) = (x_2 \leq x_1) / (b - a)$. this defines the probabilities of outcomes between x_1 and x_2.

Example: Continuous uniform distribution

X is uniformly distribution between 2 and 12. Calculate the probability that X will be between 4 and 8.

Distributions

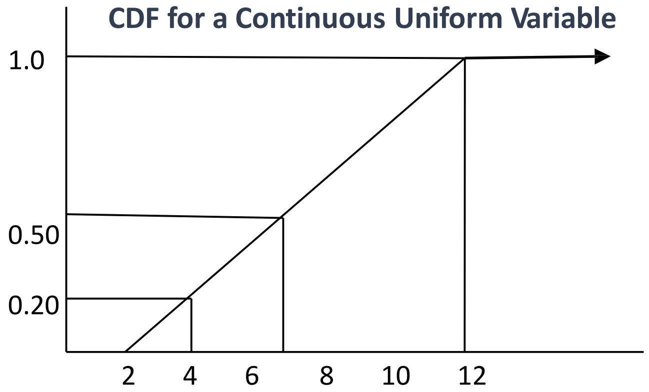
➢ **THE UNIFORM DISTRIBUTION:**

The probability function for a continuous random variable is called the probability density function (pdf) and is denoted f(x). Symbolically, the probability density function for a continuous uniform distribution is expressed as:

$$f(x)\frac{1}{b-a} for\ a \le x \le b, else\ f(x) = 0.$$

The mean and variances, respectively, of a uniform distribution are:

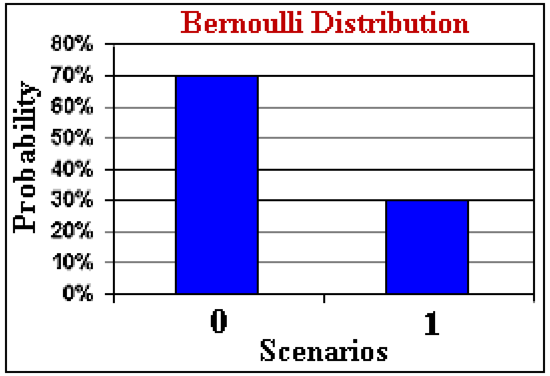
$$E(x) = \frac{a+b}{2} \qquad Var(x) = \frac{(b-a)^2}{12}$$



Distributions

➢ **THE BERNOULLI DISTRIBUTION:**

A Bernoulli distributed random variable only has two possible outcomes. The outcomes can be defined as either a "success" or a "failure." The probability of success, p, may be denoted with the value "1" and the probability of failure, 1-p, may be denoted with the value "0."



Distributions

➢ **THE BINOMIAL DISTRIBUTION:**

- ➢A binomial random variable may be defined as the number of "success".
- ➢In a given number of trials, whereby the outcome can be either "success" or "failure".
- ➢The probability of success, p, is constant for each trial and the trials are independent.
- ➢The binomial probability function defines the probability of x successes in n trials.
- ➢It can be expressed using the following formula:

$$P(x) = {}_{x}^{n}C * p^{x} * (1-p)^{n-x}$$

Example: Binomial probability

Assuming a binomial distribution, compute the probability of drawing three black beans from a bowl of black and white beans if the probability of deflecting a black in any given attempt is 0.6. You will draw five beans from the bowl.

Distributions

- **THE BINOMIAL DISTRIBUTION:**
  - **Expected Value and Variance of a Binomial Random variable**
  - For a given series of n trials, the expected number of successes is

    Expected value of X = E(X) = np
  - The variance of a binomial random variable is given by:

    Variance of X = np (1 – p) =npq

Expected: Expected value of a binomial random variable

Based on empirical data, the probability that the Dow Jones Average (DJLA) will increase eon any given day has been determine to equal 0.67. Assuming the only other outcome is that it decrease, we can state p(UP) = 0.67 and p(DOWN) = 0.33. Further, assume that movements in the DJIA are independent (i.e., an increase in one day is independent of what happened on another day). Using the information provided, compute the expected value of the number of up days in a 5-day period.

Distributions

➢ THE POISSON DISTRIBUTION:

➢The exponential function can be expressed as a series expansion. For example:

$$e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \cdots$$

➢We can use this to get a discrete random variable. Dividing both sides by e^{λ} we get:

$$1 = e^{-\lambda} + \lambda e^{-\lambda} + \frac{\lambda^2}{2!}e^{-\lambda} + \frac{\lambda^3}{3!}e^{-\lambda} + \cdots$$

➢This is the same as:

$$1 = \frac{\lambda^0}{0!}e^{-\lambda} + \frac{\lambda^1}{1!}e^{-\lambda} + \frac{\lambda^2}{2!}e^{-\lambda} + \frac{\lambda^3}{3!}e^{-\lambda} + \cdots$$

➢Since the terms add up to 1 (and are all positive) they could be our probabilities! We

➢just need to write out a general expression for each term. This is: $\frac{\lambda^x}{x!}e^{-\lambda}$ where the *x can take the values 0, 1, 2, 3, …*

➢So we have:

$$P(X = x) = \frac{\lambda^x}{x!}e^{-\lambda} \qquad x = 0, 1, 2, 3, \ldots$$

➢This is called the **Poisson distribution (after its creator Siméon Poisson).**

➢It is used mainly for modeling events, where event occur at a rate of λ per unit of time or length.

➢The shortcut way of writing *' X has a Poisson distribution with parameter λ ' is:* $X \sim Poi(\lambda)$

➢The expected value & variance of Poisson Distribution is λ

Distributions

➢ **THE POISSON DISTRIBUTION:**

Example: Using the Poisson distribution (1)

On average, the 911 emergency switchboards receive 0.1 incoming calls per second. What is the probability that in a given minute exactly 5.0 phone calls will be received, assuming the arrival of calls follows a Poisson distribution?

Example: Using the Poisson distribution (2)

Assume there is a 0.01 probability of a patient experiencing severe weight loss as a side effect from taking a recently approved drug used to treat heart disease. What is the probability that out of 200 such procedures conducted on different patients, five patients will develop this complication? Assume that the number of patients developing the complication from the procedure is Poisson- distributed.

Distributions

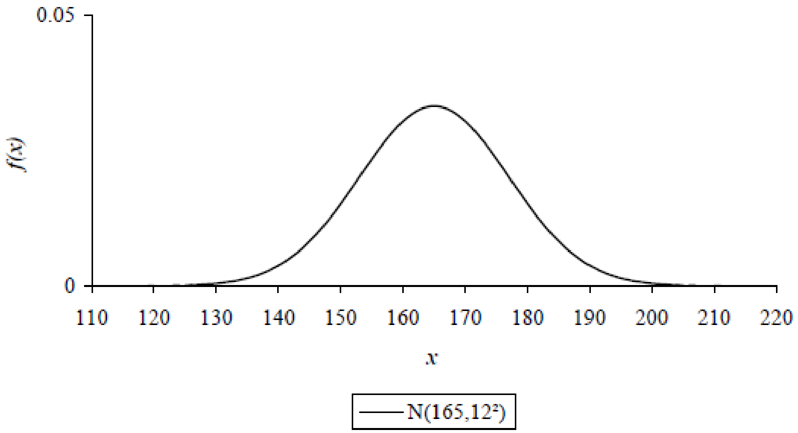
➢ THE NORMAL DISTRIBUTION:

➢The probability density function for the normal distribution is:

$$f(x) = \frac{1}{\sigma\sqrt{2\mu}} e^{-\frac{1}{2}(\frac{x-\mu}{\sigma})^2}$$

The normal distribution has the following key properties:

1. It is completely described by its mean μ and variance σ^2 stated as **X ~ N(μ ,σ^2)**
2. Skewness =0
3. Kurtosis =3
4. The expected value of mean & variance of normal distribution is **μ & σ^2**
5. A linear combination of the normally distributed independent random variable is also normally distributed.
6. The probabilities of outcomes further above below the mean get smaller and smaller but do not go to zero (the tails get very thin but extend infinitely).

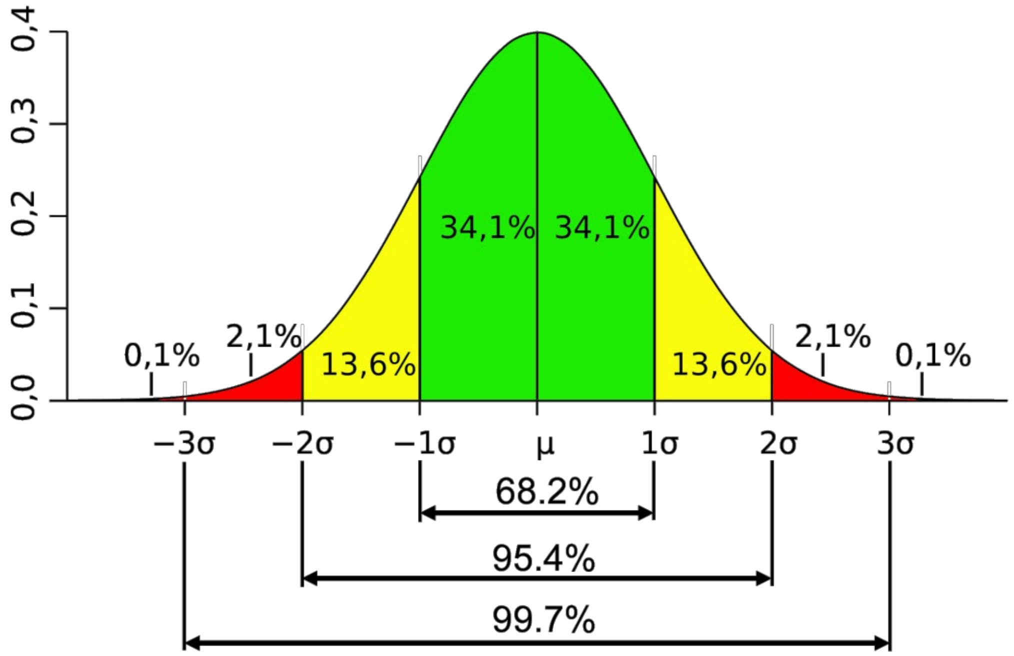


$$-\infty < x < +\infty$$

Distributions

➢ THE NORMAL DISTRIBUTION:

Confidence Intervals for a Normal Distribution



- The 90% confidence interval for $\bar{x}$ is $\bar{x} \pm 1.65s$
- The 95% confidence interval for $\bar{x}$ is $\bar{x} \pm 1.96s$
- The 99% confidence interval for $\bar{x}$ is $\bar{x} \pm 2.58s$

If $X \sim N(\mu, \sigma^2)$ then $aX + b \sim N(a\mu + b, a^2\sigma^2)$.

Distributions

➢ **THE NORMAL DISTRIBUTION:**

Example: confidence intervals

The average returns of a mutual fund are 10.5% per year and the standard deviation of annual returns is 18%. If returns are approximately normal, what is the 95% confidence interval for the mutual fund return next year?

Distributions

➢ **THE STANDARD NORMAL DISTRIBUTION:**

➢A standard normal distribution (i.e., z-distribution) is a normal distribution that has been standardized so it has a mean of zero and a standard deviation of 1 [i.e., N ~ (0,1)].

➢The z-value represents the number of standard deviations a given observation is from the population mean.

➢Standardization is the process of converting an observed value for a random variable to its z-value.

➢The following formula is used to standardize a random variable:

$$Z = \frac{observation - population\,mean}{standard\ deviation} = \frac{x - \mu}{\sigma}$$

Distributions

- **THE STANDARD NORMAL DISTRIBUTION:**

Example: Standardizing a random variable (calculating z-values)
Assume the annual earnings per share (EPS) for a population of firms are normally distribution with a mean of $6 and a standard deviation of $2.
What are the z-values for EPS of $2 and $8?

Example: Using the z-table (1)
Considering again EPS distributed with μ = $6 and σ^2 = $2, what is the probability that EPS will be $9.70 or more?

Example: Using the z-table (2)
Using the distribution of EPS with μ = $6 and σ^2 = $2 again, what percent of the observed EPS values are likely to be less than $4.10?

Distributions

➢ THE LOG NORMAL DISTRIBUTION:

➢The lognormal distribution is generated by the function e^x, where x is normally distributed.

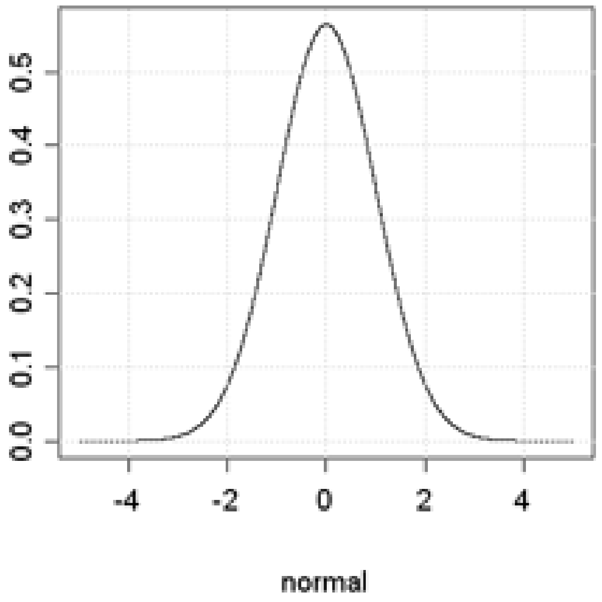
➢Since the natural logarithm, ln of e^x is x, the logarithms of log normally distributed random variables are normally distributed, thus the name.

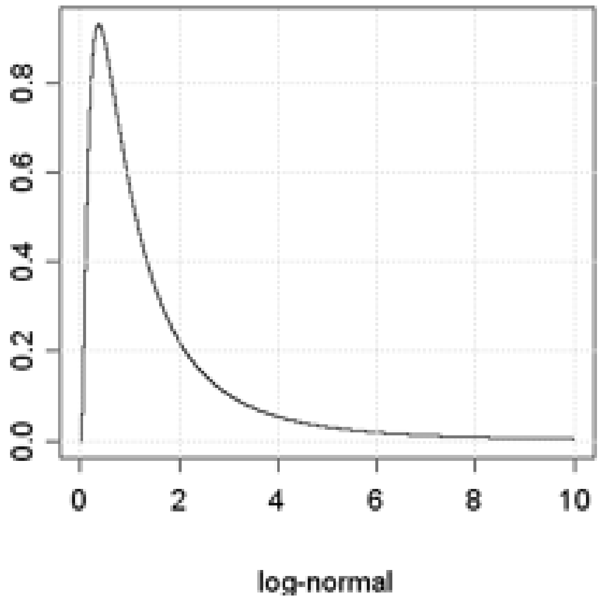
➢ The probability density function for the lognormal distribution is:

$$f(x) = \frac{1}{x\sigma\sqrt{2\mu}} e^{-\frac{1}{2}\left(\frac{\ln x - \mu}{\sigma}\right)^2}$$

➢The lognormal distribution is skewed to the right.

➢The lognormal distribution is bounded from below by zero so that it is useful for modeling asset prices which never take negative values.





Distributions

- **THE CENTRAL LIMIT THEOREM:**

➢The central limit theorem states that for simple random for simple random samples of size n from a population with a mean μ and a finite variance σ^2 the sampling distribution of the sample mean $\bar{x}$ approaches a normal probability distribution with mean μ and variance equal to $\frac{\sigma^2}{2}$ as the sample size becomes larger.

➢As long as the sample size is "sufficiently large," which usually means n $\geq$ 30, population mean can be made from the sample mean.

Properties of the central limit theorem:

1. If the sample size n is sufficiently large (n $\geq$ 30), the sampling distribution of the sample means will be approximately normal.
2. The mean of the population, and the mean of the distribution of all possible sample means are equal.
3. The variance of the distribution of sample means is population variance divided by the sample size.

Distributions

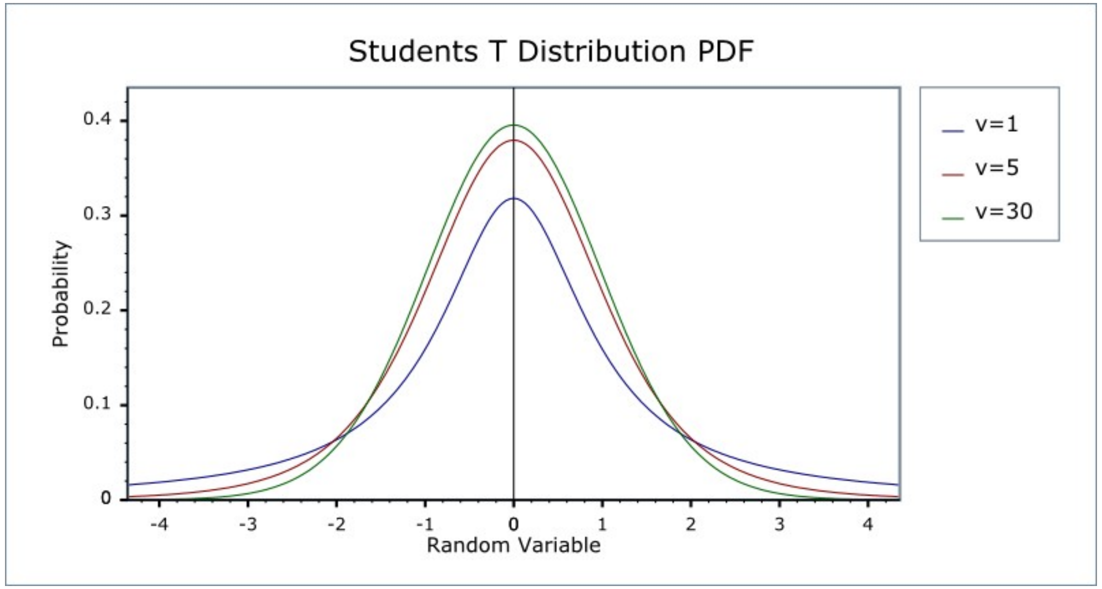
➢ STUDENT'S t-DISTRIBUTION:

➢It is the appropriate distribution to use when constructing confidence intervals based on small samples ($n < 30$) from population with unknown variance and a normal, or approximately normal, distribution.

➢It may also be appropriate to use the t-distribution when the population variance is unknown and the sample size is large enough that the central limit theorem will assure that the sampling distribution is approximately normal.

Properties:

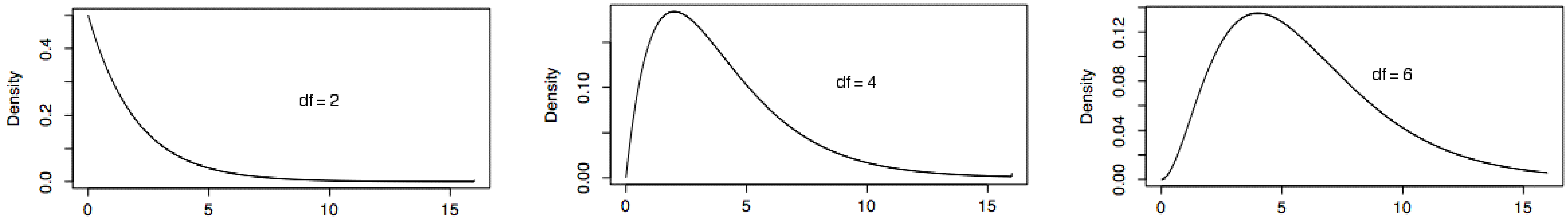
1. It is symmetrical.
2. It is defined by a single parameter, the degrees of freedom (df), where the degrees of freedom are equal to $n - 1$ for sample means.
3. It has more probability in the tails (fatter tails) that the normal distribution.
4. As the degrees of freedom (the sample size) get larger, the shape of the t-distribution closely approaches a standard normal distribution.



Distributions

➢ THE CHI –SQUARED DISTRIBUTION:

- ➢A standard normal deviate is a random sample from the *standard normal distribution*.
- ➢The Chi Square distribution is the distribution of the sum of squared *standard normal deviates*.
- ➢The *degrees of freedom* of the distribution is equal to the number of standard normal deviates being summed.
- ➢Therefore, Chi Square with one degree of freedom, written as $\chi^2(1)$, is simply the distribution of a single normal deviate squared.
- ➢The mean of a Chi Square distribution is its degrees of freedom.
- ➢Chi Square distributions are positively skewed, with the degree of skew decreasing with increasing degrees of freedom.
- ➢As the degrees of freedom increases, the Chi Square distribution approaches a normal distribution.



➢The chi-squared test statistic, χ^2, with n – 1 degrees of freedom, is computed as: $\chi^2_{n-1} = \frac{(n-1)s^2}{\sigma_0^2}$

Where: N = sample size, s^2 = sample variance, σ_0^2 = hypothesized value for the population variance

Distributions

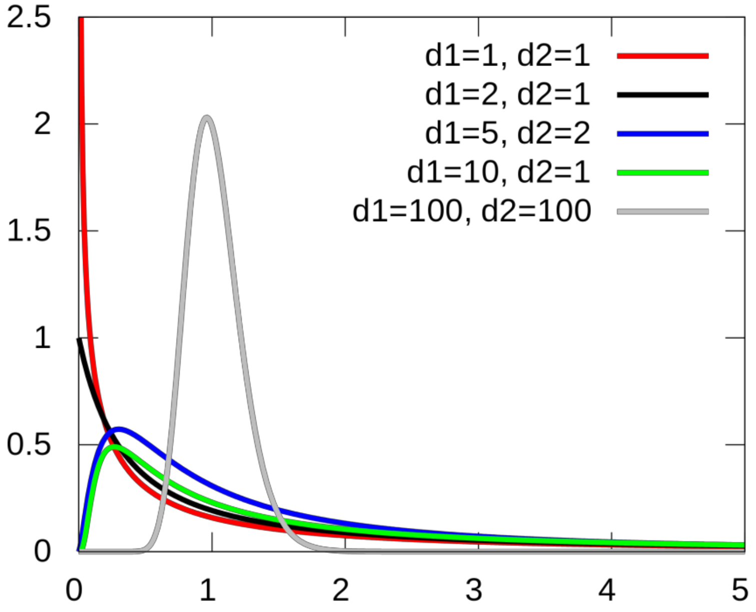
➢ THE F – DISTRIBUTION:

- ➢The equality of the variances of two populations are tested with an F-distribution test statistic.
- ➢The F-test is used under the assumption that the population from which samples are drawn are normally distributed and that the samples are independent.
- ➢The test statistic for the F-test is the ratio of the sample variances.
- ➢The shape of the F-distribution is determined by two separate degree of freedom, the numerator degrees of freedom, df_1, and the denominator degrees of freedom, df_2
- ➢The F-distribution is either zero or positive, so there are no negative values for *F*.
- ➢The F-distribution is skewed to the right.
- ➢The F-statistic is computed as: $F = \frac{s_1^2}{s_2^2}$

Where :

s_1^2 =variance of the sample of n_1 observations drawn from Population 1

s_2^2 =variance of the sample of n_2 observation drawn from Population 2



There exists a relationship between the F-and chi-squared distributions such as that :

$$F = \frac{\chi^2}{\#\ of\ observation\ in\ numertaor}$$

Distributions

➢ **MIXTURE DISTRIBUTIONS:**

➢A **mixture distribution** is the probability distribution of a random variable that is derived from a collection of other random variables as follows.

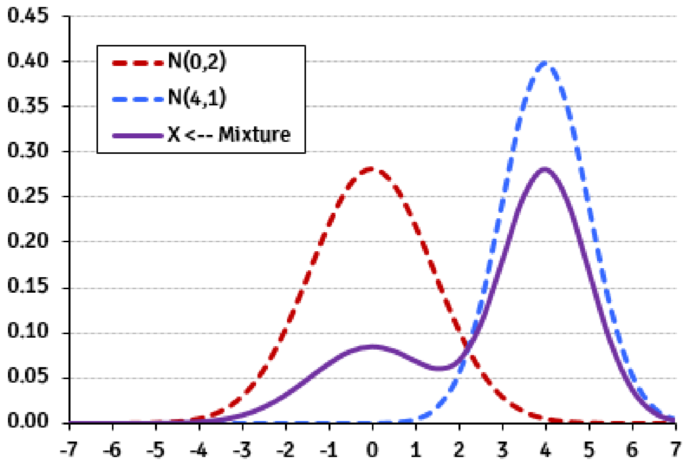
➢Mixture distributions contain elements of both parametric and nonparametric distribution.

➢The distributions used as input (i.e., the component distributions) are parametric, while the weights of each distribution within the mixture are nonparametric.

➢The more component distributions used as inputs, the more closely the mixture distribution will follow the actual data.

➢By mixing distribution, it is easy to see how we can alter skewness and kurtosis of the component distributions. Skewness can be changed by combining distributions with different means, and kurtosis can be changed by combining distributions with different variances.

➢Also, by combing distributions that have significantly different means, we can create a mixture distribution with multiple modes (e.g., a bimodal distribution).



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